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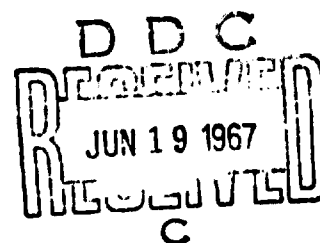
## **Aerospace Research Laboratories**

### **THE ANALYSIS OF NOISY NON-LINEAR DEVICES BY MEANS OF GENERALIZED RANDOM PROCESSES**

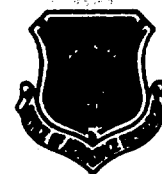
DONN G. SHANKLAND  
GENERAL PHYSICS RESEARCH LABORATORY

Project No. 7114

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**THE ANALYSIS OF NOISY NON-LINEAR DEVICES BY  
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**AEROSPACE RESEARCH LABORATORIES  
OFFICE OF AEROSPACE RESEARCH  
UNITED STATES AIR FORCE  
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## FOREWORD

This report arose from an investigation into the noise properties of optical correlation detectors as described originally by T. M. Chen and A. Van der Ziel.<sup>1</sup> The techniques were developed in a series of seminars on Generalized Random Processes, given in the Solid-State Research Laboratory, ARX, in the Fall and Winter of 1966-1967.

## ABSTRACT

The mathematical description of noise by means of Generalized Random Processes is presented. The effects on the noise distribution of linear filters (amplifiers) is discussed and related to conventional filter theory. The modification of the noise by a quadratic device is then treated, and this formalism is then applied to the analysis of the noise performance of a correlation detector and a conventional square-law detector. The conventional detector is shown to have a superior signal-to-noise ratio.

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## SECTION I

### INTRODUCTION

All instruments are ultimately limited in their sensitivity by noise, and it frequently is desirable to know quantitatively the effect of that noise. Furthermore, one is frequently interested in the probable outcome of a measurement made on an ensemble of events, i. e., the description of the output of an instrument when the input can have a range of forms with some probability distribution. It is the purpose of this report to present a most powerful formalism, due to I. M. Gel'fand and N. Ya. Vilenkin<sup>2</sup>, in a form suitable for application to these problems. The basic concepts will be developed first, then applied to linear devices. This will provide a description of both the device and the noise or signals passing through it. Then the formalism will be applied to the computation of the noise output of nonlinear (quadratic) devices, such as are commonly used for detection and power measurement. Finally, the entire procedure will be applied to a specific problem: the comparison of a quadratic and a correlation detector. This will illustrate the technique and, incidentally, demonstrate the superiority of the simpler quadratic detector.

## SECTION II

### THE GENERALIZED RANDOM PROCESS

Fundamental to the formalism is the notion of a random variable, which we define as follows: A random variable  $\xi$  is defined whenever we are given a function  $P_\xi(x)$ , where

$$P_\xi(x) = \text{Prob.} \{ \xi < x \}$$

and where

$$P_\xi(x_1) \leq P_\xi(x_2) \quad \text{if} \quad x_1 \leq x_2$$

$$\lim_{x \rightarrow -\infty} P_\xi(x) = 0, \quad \lim_{x \rightarrow +\infty} P_\xi(x) = 1$$

$$\lim_{x \rightarrow a-0} P_\xi(x) = P_\xi(a).$$

Several random variables  $\xi_1, \dots, \xi_n$  or equivalently an n-dimensional random variable  $\xi = (\xi_1, \dots, \xi_n)$  is defined by the joint distribution function

$P_\xi(x_1, \dots, x_n) = \text{Prob} \{ \xi_1 < x_1, \dots, \xi_n < x_n \}$ . A function of a random variable  $\eta = f(\xi)$  is defined as follows: Let  $X$  be the set of all points such that  $f(x) < y$  for  $x \in X$ . Then  $P_\eta(y) = P_\xi(X)$  is the distribution function for  $\eta$ . Joint distributions such as  $\text{Prob} \{ \eta < y, \xi < x \}$  are explained more fully by Gel'fand.<sup>1</sup> The moments of a random variable are the values

$$\mu_n = \int x^n dP_\xi(x) = E \{ \xi^n \},$$

where the last symbol is called the expectation of the argument.  $\mu_1$  is called the mean, and  $\mu_2$  the variance of  $\xi$ .

The Generalized Random Process is now definable as a mapping  $\Phi$  from a certain function space  $K$  into the set of random variables, and possessing the following properties: Denote the image of  $\varphi(x)$  by  $\Phi(\varphi)$ . Then

$$a. \quad \Phi(\alpha\varphi + \beta\psi) = \alpha\Phi(\varphi) + \beta\Phi(\psi) \quad (\text{Linearity})$$

$$b. \quad \lim_{n \rightarrow \infty} (\varphi_n(x), \dots, \varphi_{n+m}(x)) = (\varphi_1(x), \dots, \varphi_m(x)) \text{ implies}$$

$$\lim_{n \rightarrow \infty} (\Phi(\varphi_n), \dots, \Phi(\varphi_{n+m})) = (\Phi(\varphi_1), \dots, \Phi(\varphi_m)) \quad (\text{continuity}).$$

I.e., a Generalized Random Process is a continuous linear random functional on the space  $K$  of infinitely differentiable functions  $\varphi(t)$  having bounded supports. The motivation is the following: if an apparatus described by a function  $\varphi(t)$  is used to measure some random process  $\Phi$ , the result is a random variable  $\Phi(\varphi)$ , characterized by both the process  $\Phi$  and the apparatus function  $\varphi(t)$ .

The principal quantities of interest are the mean  $M(\varphi)$  and the correlation functional  $B(\varphi, \psi)$  defined by

$$M(\varphi) = E\{\Phi(\varphi)\} = \int x dP_\Phi(x)$$

and

$$B(\varphi, \psi) = E\{\Phi(\varphi)\Phi(\psi)\} = \int xy dP_\Phi(x, y).$$

The most common example of a generalized random process is a Gaussian process defined by the joint distribution function

$$P_\Phi(X) = \text{Prob}\{\Phi(\varphi_1) \in X_1, \dots, \Phi(\varphi_n) \in X_n\}, \quad X = X_1 \otimes \dots \otimes X_n,$$

where

$$P_\Phi(X) = \frac{\sqrt{\det \Lambda_\varphi}}{(2\pi)^{n/2}} \int_X dx \exp\left\{-\frac{1}{2} (\Lambda_\varphi x, x)\right\}$$

and  $\Lambda_\varphi = ((\lambda_{ij}))$  is a non-degenerate positive-definite matrix, with  $(\Lambda x, x) = \sum_{i,j} \lambda_{ij} x_i x_j$ . Some properties of the distribution  $P_\Phi(X)$  are

$$a. \quad P_\Phi(R_n) = 1$$

$$b. \quad E\{\Phi(\varphi)\} = M(\varphi) = 0$$

$$c. \quad E\{\Phi(\varphi_i)\Phi(\varphi_j)\} = B(\varphi_i, \varphi_j) = \Lambda^{-1}_{ij}.$$

Thus, this is a mean zero process, and is defined uniquely by the matrix  $\Lambda$  which in turn is defined by the correlation function  $B(\varphi, \psi)$ . So specifying the correlation function completely specifies the process. From the form of  $B(\varphi, \psi)$  one sees that it must be a continuous bilinear positive definite functional of its two arguments. The usual form for such a functional is

$$B(\varphi, \psi) = \int ds dt \varphi(s) \psi(t) B(s, t)$$

where  $B(s, t)$  is a positive-definite generalized function of two variables.

When  $(\Phi(\varphi_1(x)), \dots, \Phi(\varphi_n(x)))$  and

$(\Phi(\varphi_1(x+h)), \dots, \Phi(\varphi_n(x+h)))$  are identically

distributed, the process  $\Phi$  is called stationary. It can then be seen that

$$B(\varphi, \psi) = \int ds dt \varphi(s) \psi(t) B(s-t).$$

Then if  $\tilde{\varphi}(\lambda)$ ,  $\tilde{\psi}(\lambda)$ ,  $\tilde{B}(\lambda)$  are the Fourier transforms of  $\varphi(s)$ ,  $\psi(t)$ , and  $B(s-t)$  respectively, one can write

$$B(\varphi, \psi) = \int d\lambda B(\lambda) \tilde{\varphi}(\lambda) \overline{\tilde{\psi}(\lambda)},$$

where  $B(\lambda) \geq 0$ , so that  $B(\lambda) d\lambda = d\zeta(\lambda)$  is a non-negative measure. A special case is the Unit Process, defined by

$$B(s-t) = \delta(s-t), \quad \text{or} \quad d\zeta(\lambda) = d\lambda, \quad (1)$$

where  $\delta(x)$  is the Dirac delta-function.

### SECTION III

#### THE DESCRIPTION OF BAND-LIMITED NOISE

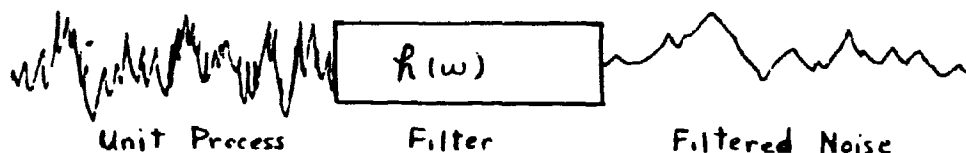


fig. 1



Assume in Fig. 1 one has a filter of transfer function

$$H_t(t') \equiv H(t-t') = \int d\omega h(\omega) e^{i\omega(t-t')},$$

so that if the input is  $v_i(t')$  the output is  $v_o(t) = \int dt' H(t-t') v_i(t')$ . Then if the input is white noise ( $\equiv$  unit process), the output at time  $t$  is a random variable  $\Phi(H_t)$  and the correlation functional is

$$B(H_s, H_t) = \int dt' H(s-t') \overline{H(t-t')} = B(s-t),$$

or

$$\begin{aligned} B(s-t) &= \int dt' d\omega d\mu h(\omega) e^{i\omega(s-t')} \overline{h(\mu) e^{i\mu(t-t')}} = \\ &= \frac{1}{2\pi} \int d\omega e^{i\omega(s-t)} |2\pi h(\omega)|^2. \end{aligned}$$

Then, if the filter consists of several sections, e. g.,

$$H(s-t) = \int dt' K(s-t') L(t'-t),$$

where

$$K(s) = \int d\omega e^{i\omega s} k(\omega); \quad L(s) = \int d\omega e^{i\omega s} l(\omega),$$

then

$$\begin{aligned} 2\pi h(\omega) &= \int du e^{-i\omega u} H(u) = \\ &= \int du dt' d\mu dv e^{-i\omega u} e^{i\nu(u+t-t')} k(\nu) e^{i\mu(t'-t)} l(\mu) = \\ &= [2\pi k(\omega)] [2\pi l(\omega)], \end{aligned}$$

so that successive linear processes can readily be introduced. Thus, for a succession of filters, we have

$$B(s, t) = \frac{1}{2\pi} \int d\omega e^{i\omega(s-t)} |2\pi k(\omega)|^2 |2\pi l(\omega)|^2 \dots |2\pi z(\omega)|^2,$$

for as many functions as are included. A useful concept is to define the "spectral power density" of the result of white-noise + filter by the spectral function

$$\rho(\omega) = |2\pi k(\omega)|^2.$$

Then successive filtering is seen to modify the spectral power density by

$$\rho^{(n+1)}(\omega) = |2\pi k^{(n+1)}(\omega)|^2 \rho^{(n)}(\omega),$$

where  $k^{(n+1)}(\omega)$  is the transfer function of the  $(n+1)$ -st filter. In this sense, the unit process is seen (from (1)) to have

$$\rho_u(\omega) \equiv 1.$$

#### SECTION IV

##### THE POWER IN A RANDOM SIGNAL

The power in a signal is proportional to the mean square of the signal, or

$$W = E\{v_c(t)v_c(t)\} = B(H_t, H_t) = B(t, t),$$

or

$$W = \frac{1}{2\pi} \int d\omega |2\pi k(\omega)|^2 = \frac{1}{2\pi} \int d\omega \rho(\omega),$$

thus motivating the term "spectral power density" for  $\rho(\omega)$ . The unit process is thus seen to have a uniform distribution of power over all frequency ranges, hence an infinite total power. Passage through a filter then limits the frequency range, giving a finite total power at the output. The idealization of white noise to the unit process introduces no error, so long as the region of uniform power distribution of the white noise is larger than the bandwidth of the filter.

#### SECTION V

##### THE DISTRIBUTION OF THE SQUARE OF A GAUSSIAN RANDOM SIGNAL

The signal is a random function  $\Phi(H_t) \equiv \xi(t)$  with a Gaussian distribution of mean zero and with correlation function

$$B(s, t) = E\{\xi(s)\xi(t)\} = \frac{1}{2\pi} \int d\omega e^{i\omega(s-t)} \rho(\omega).$$

The result of squaring is a random function  $\eta(t) = \xi^2(t)$  which is no longer Gaussianly distributed. However, as we will be interested only in the first and second moments of  $\eta(t)$ , it will suffice to replace it with a Gaussian process with the same first two moments (GV 257, corollary).<sup>\*</sup> Accordingly, we must compute

$$\bar{\eta} = E\{\eta(t)\} = E\{\xi^2(t)\} = \sqrt{\frac{|C|}{2\pi}} \int dx \ x^2 \exp\left\{-\frac{1}{2}(Cx, x)\right\},$$

where

$$C^{-1} = B(t, t) = b_0. \quad \text{We obtain } \bar{\eta} = b_0.$$

Then we must compute

$$\begin{aligned} D(s, t) &\equiv E\{\eta(s)\eta(t)\} = E\{\xi^2(s)\xi^2(t)\} = \\ &= \sqrt{\frac{|C|}{2\pi}} \int dx dy \ x^2 y^2 \exp\left\{-\frac{1}{2}(Cx, x)\right\}, \end{aligned}$$

where

$$C^{-1} = \begin{bmatrix} B(s, s) & B(s, t) \\ B(t, s) & B(t, t) \end{bmatrix}. \quad \text{Let } B(s, t) = b_{st}.$$

In this and subsequent calculations we will need an extension of the calculation of Gel'fand (GV 250, Eq. 5), or

$$\frac{\sqrt{|C|}}{(2\pi)^{n/2}} \int dx \ (Ax, x) \exp\left\{-\frac{1}{2}(Cx, x)\right\} = \text{Tr}(AC^{-1}) = a_1, \quad (2)$$

$$\frac{\sqrt{|C|}}{(2\pi)^{n/2}} \int dx \ (Ax, x)^2 \exp\left\{-\frac{1}{2}(Cx, x)\right\} = 3a_1^2 - 4a_2, \quad (3)$$

<sup>\*</sup>References are to page numbers in Gel'fand and Vilenkin.<sup>2</sup>

where  $a_2$  is the sum of all the principal  $2 \times 2$  minors of  $AC^{-1}$ . For our problem,  $(Ax, x) = (xy + yx)/2$ , or

$$A = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{ Then } AC^{-1} = AB = \frac{1}{2} \begin{bmatrix} b_{sr} & b_c \\ b_c & b_{sr} \end{bmatrix},$$

$$a_1 = b_{sr}, \quad a_2 = \frac{1}{4} (b_{sr}^2 - b_c^2).$$

Then from (3),

$$E\{\eta(s)\eta(t)\} = D(s,t) = 2B^2(s,t) + b_c^2.$$

The power density spectrum of this process is then

$$\begin{aligned} \rho_2(\omega) &= \int du e^{-i\omega u} D(u) = \\ &= \frac{2}{(2\pi)^2} \int d\mu d\nu e^{-i\omega u} e^{i\mu u} \rho(\mu) e^{-i\nu u} \overline{\rho(\nu)} + 2\pi b_c^2 \delta(\omega) = \\ &= \frac{2}{2\pi} \int d\mu \rho(\mu) \overline{\rho(\mu-\omega)} + 2\pi b_c^2 \delta(\omega). \end{aligned}$$

(4)

The  $S(\omega)$  term arises from the DC power of the non-zero mean. If we considered instead the process  $\eta(s) - \bar{\eta}$ , we have a zero mean process with

$$E\{(\eta(s) - \bar{\eta})(\eta(t) - \bar{\eta})\} = E\{\eta(s)\eta(t)\} - \bar{\eta}^2 = 2B^2(s,t),$$

and

$$\rho_2(\omega) = \frac{2}{2\pi} \int d\mu \rho(\mu) \overline{\rho(\mu-\omega)}.$$

If the squaring is followed by additional filtering (e. g., integration), then the final signal has a mean

$$\bar{\eta}_f = |2\pi f(\omega)| \bar{\eta} = |2\pi f(\omega)| b_c,$$

and a correlation function

$$\begin{aligned} D_f(s, t) &= \frac{1}{2\pi} \int d\omega e^{i\omega(s-t)} |2\pi f(\omega)|^2 \rho(\omega) = \\ &= \frac{1}{2\pi} \int d\omega e^{i\omega(s-t)} |2\pi f(\omega)|^2 \left\{ \frac{2}{2\pi} \int d\mu \rho(\mu) \overline{\rho(\mu-\omega)} \right\}, \end{aligned} \quad (5)$$

where  $f(\omega)$  is the final filter function.

## SECTION VI

### THE DISTRIBUTION OF THE PRODUCT OF TWO GAUSSIAN RANDOM SIGNALS

Consider two Gaussianly distributed random signals  $\xi_1(s) = \xi(s) + \eta_1(s)$  and  $\xi_2(s) = \xi(s) + \eta_2(s)$ , where  $\eta_1(s)$  and  $\eta_2(s)$  are independent but identically distributed zero-mean Gaussian signals with correlation function

$$E\{\eta_1(s)\eta_1(t)\} = B(s-t) = E\{\eta_2(s)\eta_2(t)\},$$

and where  $\xi(s)$  is a common signal, independent of  $\eta_1(s)$  and  $\eta_2(s)$  and with correlation function

$$E\{\xi(s)\xi(t)\} = B'(s, t).$$

We wish to compute the first two moments of the product of  $\xi_1(s)$  and  $\xi_2(s)$  in order to define an equivalent Gaussian process, as we did in Section V. With  $\eta(s) = \xi_1(s)\xi_2(s)$  we need  $E\{\eta(s)\} = E\{\xi_1(s)\xi_2(s)\} = \bar{\eta}$  and

$$E\{\eta(s)\eta(t)\} = E\{\xi_1(s)\xi_2(s)\xi_1(t)\xi_2(t)\} = D(s, t).$$

To compute these, we need the correlation functions

$$E\{\xi_1(s)\xi_1(t)\} = E\{(\xi(s) + \eta_1(s))(\xi(t) + \eta_1(t))\} = E\{\xi(s)\xi(t)\} = B'(s, t),$$

and

$$E\{\xi_1(s)\xi_2(t)\} = E\{(\xi_1(s) + \eta_1(s))(\xi_2(t) + \eta_2(t))\} = E\{\xi_1(s)\xi_2(t)\} = B(s,t) + B'(s,t).$$

Thus, if we let  $(x_1, x_2, x_3, x_4)$

be the observed values of

$(\xi_1(s), \xi_2(t), \xi_1(s), \xi_2(t))$ ,

the relevant correlation matrix is

$$B = \begin{bmatrix} b'_0 + b_c & b'_{st} + b_{sr} & b'_0 & b'_{st} \\ b'_{st} + b_{sr} & b'_0 + b_c & b'_{st} & b'_0 \\ b'_0 & b'_{st} & b'_0 + b_c & b'_{st} + b_{sr} \\ b'_{st} & b'_0 & b'_{st} + b_{sr} & b'_0 + b_c \end{bmatrix} = C^{-1}.$$

To compute the mean  $\bar{\eta} = E\{\eta(s)\} = E\left\{\frac{\xi_1(s)\xi_2(s) + \xi_2(s)\xi_1(s)}{2}\right\}$ ,  
use Eq. (2), where

we

$$A = \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so } \bar{\eta} = \text{Tr} AC^{-1} = b'_0.$$

To compute the correlation function, we have

$$D(s,t) = E\{\xi_1(s)\xi_2(s)\xi_2(t)\xi_1(t)\} =$$

$$= \frac{\sqrt{|C|}}{(2\pi)^2} \int dx_1 dx_2 dx_3 dx_4 (x_1 x_2 x_3 x_4) \exp\left\{-\frac{1}{2} (Cx, x)\right\}.$$

By making the orthogonal transformation

$$\underline{x} = \underline{U} * \underline{y} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} * \underline{y},$$

we have

$$(C\underline{x}, \underline{x}) = \tilde{\underline{x}} * \underline{C} * \underline{x} = \tilde{\underline{y}} * \tilde{\underline{U}} * \underline{C} * \underline{U} * \underline{y} = \tilde{\underline{y}} * (\underline{U} * \underline{B} * \underline{U})^{-1} * \underline{y},$$

where

$$(\underline{U} * \underline{B} * \underline{U}) = \begin{bmatrix} 2b_c' + b_0 & 2b_{st}' + b_{st} & 0 & 0 \\ 2b_{st}' + b_{st} & 2b_o' + b_0 & 0 & 0 \\ 0 & 0 & b_c & b_{st} \\ 0 & 0 & b_{st} & b_o \end{bmatrix}.$$

Thus, the components  $(y_1, y_2)$  are independent of the components  $(y_3, y_4)$ . Furthermore,

$$x_1 x_3 x_3 x_4 = (y_1^2 - y_3^2)(y_2^2 - y_4^2)/4.$$

We obviously have

$$E\{y_1^2\} = E\{y_2^2\} = 2b_o' + b_0; \quad E\{y_3^2\} = E\{y_4^2\} = b_0;$$

and for  $E\{y_3^2 y_4^2\}$ , we use (3), where

$$A = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$E\{y_3^2 y_4^2\} = 3b_{st}^2 - 4[(b_{st}^2 - b_c^2)/4] = 2b_{st}^2 + b_c^2,$$

and similarly,

$$E\{y_1^2 y_2^2\} = 2(b_{st}' + b_{st})^2 + (2b_0' + b_0)^2$$

From these, we get

$$E\{\eta(s)\eta(t)\} = D(s,t) = b_{st}^2 + 2b_{st}b_{st}' + 2b_{st}'^2 + b_0'^2.$$

The power density spectrum of this process is then, like in (4),

$$\rho_3(\omega) = \frac{1}{2\pi} \int d\mu [\rho(\mu)\overline{\rho(\mu-\omega)} + 2\rho(\mu)\overline{\rho'(\mu-\omega)} + 2\rho'(\mu)\overline{\rho'(\mu-\omega)}] + 2\pi b_0'^2 \delta(\omega).$$

Further, after subtracting the mean  $b_0'$ , we have

$$D(s,t) = b_{st}^2 + 2b_{st}b_{st}' + 2b_{st}'^2,$$

and after final filtering,

$$\rho_3(\omega) = \frac{1}{2\pi} \int d\mu [\rho(\mu)\overline{\rho(\mu-\omega)} + 2\rho(\mu)\overline{\rho'(\mu-\omega)} + 2\rho'(\mu)\overline{\rho'(\mu-\omega)}] |2\pi f(\omega)|^2. \quad (6)$$

## SECTION VII

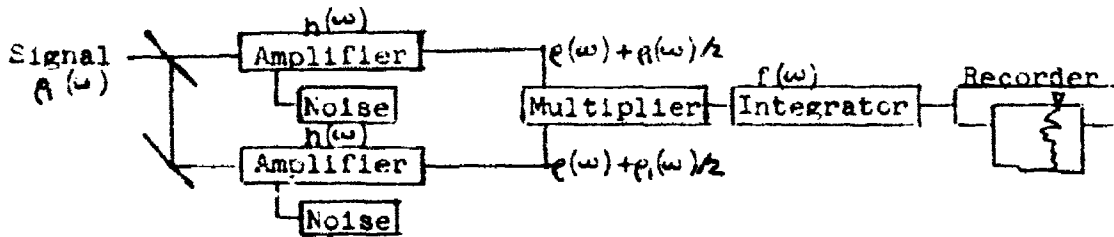
### A COMPARISON OF QUADRATIC AND CORRELATION DETECTORS

The block diagrams of the two assumed experimental arrangements are given in Fig. 2. In the actual measurement, what is ordinarily recorded is the actual signal at the output of the integrator. So the quantities of interest are the mean of the (signal + noise) output and its variation, as compared to the noise-only output. It is presumed that the equipment is sufficiently stable so that a good determination of the noise-only means can be made. These means can be subtracted from the output, e. g., by offsetting the recorder pen. Then both outputs have mean zero in the absence of signal, and have rms deviations  $\sigma_{CN}$  for the correlation detector (CD) and  $\sigma_{QN}$  for the quadratic detector (QD) as given by (6) and (5) with  $\rho'(\omega) = 0$ ,

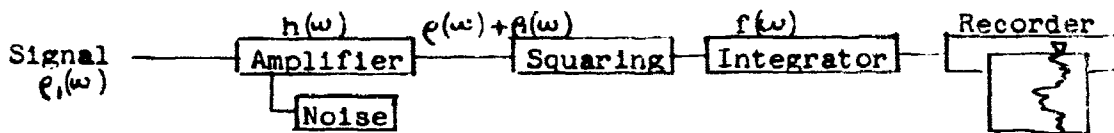
$$\sigma_{CN}^2 = D_f(s,s) = \frac{1}{2\pi} \int d\omega |2\pi f(\omega)|^2 \left\{ \frac{1}{2\pi} \int d\mu \rho(\mu)\overline{\rho(\mu-\omega)} \right\},$$



$$\sigma_{en}^2 = D_f(s, s) = \frac{1}{2\pi} \int d\omega |2\pi f(\omega)|^2 \left\{ \frac{2}{2\pi} \int d\mu \rho(\mu) \overline{\rho(\mu-\omega)} \right\} = 2\sigma_{CN}^2.$$



Correlation Detector (CD)



Quadratic Detector (QD)

fig. 2.

Thus, the noise-only variation in the QD is twice the variation in the CD. However, the signal input power is split between the two channels in the CD, so that with the signal on, we must compare the results of a  $p_i(\omega)/2$  input to each channel of the CD, while we have a  $p_i(\omega)$  input to the QD.

The desired knowledge of the input signal is its mean power, as evidenced by the shift in the mean of the output of the detectors when the signal is on. Thus, when we substitute  $p(\omega) + p'(\omega)$  for  $p(\omega)$  in (4) and (5), we get

$$\Delta \bar{\eta}_Q = \frac{1}{2\pi} \int d\omega p'(\omega) = b_0',$$

and a variation for the  $\eta(s)$  about its new mean of

$$\sigma_{Q(s+n)}^2 = \frac{1}{2\pi} \int d\omega |2\pi f(\omega)|^2 \left\{ \frac{2}{2\pi} \int d\mu [p(\mu) \overline{p(\mu-\omega)} + 2p(\mu) \overline{p'(\mu-\omega)} + p'(\mu) \overline{p'(\mu-\omega)}] \right\}.$$

For the CD, when the input signal to each channel is  $e'(\omega)/2$ , we get

$$\Delta \bar{\eta}_c = \frac{1}{2} \int d\omega e'(\omega) = b'_0/2,$$

and (6) gives a variation about the new mean of

$$\sigma_{c(s.m)}^2 = \frac{1}{2\pi} \int d\omega |2\pi f(\omega)|^2 \left\{ \frac{1}{2\pi} \int d\mu [e(\mu) \overline{e(\mu-\omega)} + e(\mu) \overline{e'(\mu-\omega)} + \frac{1}{2} e'(\mu) \overline{e'(\mu-\omega)}] \right\}.$$

In both of the arrangements, the amplifier noise is regarded as white noise limited by the amplifier response, so  $e(\omega) = |2\pi h(\omega)|^2$ . The signal was assumed to lie entirely within the amplifier bandwidth, or alternatively  $e'(\omega)$  can be considered as the signal power density spectrum  $e_s(\omega)$  multiplied by the square of the amplifier transfer function,

$$e'(\omega) = |2\pi h(\omega)|^2 e_s(\omega).$$

The signal to noise ratio of the quadratic detector,

$$R_Q = \frac{\Delta \bar{\eta}_e}{\sqrt{\sigma_{eN}^2 + \sigma_{e(s.m)}^2}}$$

and the similar expression for the correlation detector

$$R_c = \frac{\Delta \bar{\eta}_c}{\sqrt{\sigma_{cN}^2 + \sigma_{c(s.m)}^2}}$$

can be related in the critical case where  $e(\omega) \gg e'(\omega)$ , since there  $\sigma_{s.m}^2 \rightarrow \sigma_N^2$  and

$$\frac{R_Q}{R_c} = \left( \frac{b'_0}{\sigma_Q \sqrt{2}} \right) / \left( \frac{b'_0}{2 \sigma_c \sqrt{2}} \right) = \sqrt{2},$$

I.e., the signal-to-noise ratio of the quadratic detector is  $\sqrt{2}$  times as large as the signal-to-noise ratio of the correlation detector.

These values can be easily obtained explicitly in the case where  $e'(\omega) = 2\pi \alpha \delta(\omega - \omega_0)$ , and the final filter is an integrator

$$v_c(t) = \frac{1}{2\pi} \int_{t-T}^{t+T} dt' v_i(t')$$

with transfer function  $|2\pi f(\omega)| = \sin(\omega T)/(\omega T)$ . This is the case of a monochromatic signal of unknown amplitude incident on a noisy receiver. Then

$$\Delta \bar{\eta}_Q = a = 2\Delta \bar{\eta}_C,$$

$$\sigma_{QN}^2 = \frac{1}{2\pi T} \int d\mu \rho(\mu) \overline{\rho(\mu)} = 2\sigma_N^2,$$

$$\sigma_{Q(S+N)}^2 = \sigma_{QN}^2 + \frac{2a\rho(\omega_0)}{T} + 2a^2$$

$$\sigma_{C(S+N)}^2 = \sigma_{CN}^2 + \frac{a\rho(\omega_0)}{2T} + \frac{a^2}{2}$$

Thus, as  $T \rightarrow \infty$ , the dispersion is due to the dispersion in the quantities being measured. For large  $T$ , furthermore, the ratio of output to dispersion becomes the same for both devices, while for finite  $T$  and small signals it favors the quadratic detector.

## SECTION VIII

### SUMMARY

A procedure has been presented for analyzing the effect of any linear or nonlinear device upon a Gaussianly distributed random signal. The signal was seen to be representable in terms of its spectral power density  $\rho(\omega)$ , the power per unit frequency interval. The effect of any linear device was rigorously shown to be the multiplication of the spectral power density by

$$|2\pi h(\omega)|^2,$$

where  $h(\omega)$  was the Fourier transform of the filter's transfer function. Furthermore, the power density in the output of any non-linear device was derived in terms of the correlation functional

$$D(s, t) = E\{\eta(s)\eta(t)\},$$

where  $\eta(s) = F(\xi(s))$  describes the non-linear transfer function. This correlation functional, in turn, could be used to define a Gaussian Generalized Random process for calculations involving subsequent filtering, so long as only the mean and second moment (power) were of interest. As an example, the method was used to analyze the signal-to-noise properties of a quadratic and a correlation detector, and somewhat surprisingly, demonstrated the superiority of the quadratic detector. The process is clearly capable of extension to other more complicated devices.

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13 ABSTRACT The mathematical description of noise by means of Generalized Random Processes is presented. The effects on the noise distribution of linear filters (amplifiers) is discussed and related to conventional filter theory. The modification of the noise by a quadratic device is then treated, and this formalism is then applied to the analysis of the noise performance of a correlation detector and a conventional square-law detector. The conventional detector is shown to have a superior signal-to-noise ratio.			

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